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Differential Calculus on Fuzzy Sphere and Scalar Field

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Abstract

We find that there is an alternative possibility to define the chirality operator on the fuzzy sphere, due to the ambiguity of the operator ordering. Adopting this new chirality operator and the corresponding Dirac operator, we define Connes' spectral triple on the fuzzy sphere and the differential calculus. The differential calculus based on this new spectral triple is simplified considerably. Using this formulation the action of the scalar field is derived.

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1 Introduction

Recent developments of the nonperturbative aspects in string theory give signs that the noncommutative geometry is the right way to describe the quantum regime of spacetime [1, 2]. Compared to other attempts in the past to generalize the spacetime structure, the noncommutative geometry has the advantage that it can preserve the kinematical properties of the geometry treated, and the framework is general enough to extend to arbitrary dimension and to include nontrivial topology.

The notion of 'noncommutative geometry' has been first considered by von Neumann. Later Connes extended this notion to the 'noncommutative differential geometry' which means that we consider the noncommutative function algebra associated with a certain geometry together with a differential calculus. Technically, it is the translation of the various geometrical concepts into the algebraic terminology, where the original space X is replaced by a function algebra $C^\infty(X)$ of the smooth functions on X. Once the geometry described by purely algebraic language, it is possible to replace its structure algebra by a noncommutative algebra and we obtain the geometry of the noncommutative space [3] .

The fuzzy sphere is one of the simplest examples of noncommutative geometry, and has been discussed by many authors. The algebra of the fuzzy sphere appeared already in many different contexts [4, 5, 6, 7, 8]. This algebra can be constructed by applying Berezin's quantization [9, 10] to the function algebra over the sphere, and the result is a matrix algebra of finite dimension. However the way to introduce the differential calculus is not unique even if we require that in the commutative limit the standard differential calculus is reproduced. Proposals for differential calculi have been made by [3, 11] and especially for the quantum sphere in [8, 12, 13].

In this paper we propose a new spectral triple for the differential calculus on the fuzzy sphere within Connes' framework. As is known, in order to apply Connes' construction of the differential algebra, it is necessary to define the Dirac operator and the chirality operator [14, 15, 16]. The approach taken here to construct the Dirac operator is similar to ref.[16]. The difference is that we adopt a different chirality operator, which is possible due to the ambiguity of operator ordering. Using this new chirality operator and the corresponding Dirac operator, we define a differential algebra within Connes' framework which has a rather simple structure. With the result-

ing differential calculus we derive the action of the scalar boson field on the noncommutative sphere.

This paper is organized as follows. In section 2, we give some definitions which are necessary to define the chirality operator and the Dirac operator on the fuzzy sphere. In section 3, the new chirality operator and Dirac operator are introduced and the differential calculus is defined using these operators. In section 4, we apply this differential calculus to the scalar field and derive the action of the scalar boson. Section 5 is the discussion.

2 Algebra of Fuzzy Sphere

Let us first recall the definition of the fuzzy sphere. The algebra $\mathcal{A}_N$ of the fuzzy sphere is generated by the operators $\mathbf{x}_i$ $(i = 1, 2, 3)$ satisfying

$$[\mathbf{x}_i, \mathbf{x}_j] = i\alpha\epsilon_{ijk}\mathbf{x}_k \ , \tag{1}$$

with the constraint

$$\mathbf{x}_i\mathbf{x}_i = \ell^2 \ . \tag{2}$$

Each operator of $\mathcal{A}_N$ is represented by a matrix acting on the $N+1$-dimensional Hilbert space $\mathcal{F}_N$. The "Planck constant" α is a central element, $[\alpha, \mathbf{x}_i] = 0$, and its value is determined by eq.(2) as $\alpha = \frac{2\ell}{\sqrt{N(N+2)}}$. α vanishes in the limit $N \to \infty$, which is the commutative limit of the algebra. Apparently, the matrices $\mathbf{x}_i$ can be identified with the generators of the $su(2)$ Lie algebra and the generated algebra is equivalent to the algebra of $(N+1)\times(N+1)$ matrices, $M_{\mathbb{C}}(N+1)$.

A natural object as a field on the fuzzy sphere is an $\mathcal{A}_N$-bimodule, where we can consider left multiplication and right multiplication of the algebra $\mathcal{A}_N$ onto this field. In the commutative limit, the distinguishing of right and left multiplication is not important, however in the noncommutative case it has rather nontrivial consequences. Since left multiplication and right multiplication commute even in the case of noncommutative space, the $\mathcal{A}_N$-bimodule can be considered as a left module over the algebra $\mathcal{A}_N \otimes \mathcal{A}_N^{\mathbf{o}}$, where $\mathcal{A}_N^{\mathbf{o}}$ denotes the opposite algebra which is defined by:

$$\mathbf{x}_i^{\mathbf{o}}\mathbf{x}_j^{\mathbf{o}} \equiv (\mathbf{x}_j\mathbf{x}_i)^{\mathbf{o}} \ , \ \mathbf{x}_i \in \mathcal{A}_N \ . \tag{3}$$

The action of $\mathbf{a}, \mathbf{b} \in \mathcal{A}_N$ onto the $\mathcal{A}_N$-bimodule $\Psi \in \Gamma\mathcal{A}_N$ is

$$\mathbf{a}\mathbf{b}^{\mathbf{o}}\,\Psi \equiv \mathbf{a}\,\Psi\,\mathbf{b} \ . \tag{4}$$

Therefore, when we consider operators acting on fields over the fuzzy sphere, these operators are elements of the algebra $\mathcal{A}_N \otimes \mathcal{A}_N^{\mathbf{o}}$ [17]. And it is natural to define a differential operator acting on the field as an element in $\mathcal{A}_N \otimes \mathcal{A}_N^{\mathbf{o}}$. As we shall see in the following, the Dirac operator derived in ref.[16] also fits to this scheme.

3 Dirac Operator and Differential Calculus

Introducing the spinor field Ψ as an $\mathcal{A}_N$-bimodule $\Gamma\mathcal{A}_N \equiv \mathbb{C}^2 \otimes \mathcal{A}_N$ we define the chirality operator and the Dirac operator $\mathbf{D}$ as operators on the spinor field, i.e. as 2×2 matrices the entries of which are elements of the algebra $\mathcal{A}_N \otimes \mathcal{A}_N^{\mathbf{o}}$. The construction of the Dirac operator can be performed by the following steps:

(a) define a chirality operator which has a standard commutative limit,

(b) define the Dirac operator requiring that it anticommutes with the chirality operator and, in the commutative limit it reproduces the standard Dirac operator on the sphere.

In ref.[16] we have constructed the Dirac operator by applying this approach. There, we have introduced a chirality operator given by

$$\gamma_\chi = \frac{1}{\mathcal{N}}(\sigma_i \otimes \mathbf{x}_i + \frac{\alpha}{2}) \ , \tag{5}$$

where $\mathcal{N}$ is a normalization constant determined by the condition $\gamma_\chi^2 = 1$, and σ_i $(i = 1, 2, 3)$ are the Pauli matrices. In the commutative limit, the operator $\mathbf{x}_i$ can be identified with the homogeneous coordinate x_i of sphere and the chirality operator given in eq.(5) becomes $\frac{\sigma_i x_i}{\ell}$ which is the standard chirality operator invariant under the rotation. We have also defined the corresponding Dirac operator. Since the chirality operator does not commute with the original algebra $\mathcal{A}_N$, we proposed a modified algebra which commutes with γ_χ, to obtain Connes' spectral triple.

In this paper, we use a different chirality operator which also produces the standard chirality operator in the commutative limit. This is possible due to the fact that the condition for the commutative limit of the chirality operator does not fix its noncommutative generalization uniquely, i.e. we have an ambiguity in the operator ordering. The new chirality operator commutes with the original algebra $\mathcal{A}_N$ and we obtain the spectral triple with the original algebra $\mathcal{A}_N$; hence, a modification of the algebra as proposed in ref.[16] is not necessary.

Requiring the same condition as for γ_χ, i.e. (a), the new chirality operator is determined as

$$\gamma_\chi^{\mathbf{o}} = \frac{1}{\mathcal{N}}(\sigma_i \otimes \mathbf{x}_i^{\mathbf{o}} - \frac{\alpha}{2}) \ . \tag{6}$$

In the following we do not explicitly write the symbol $\otimes$ since its meaning is obvious. The normalization constant $\mathcal{N}$ is defined by the condition,

$$(\gamma_\chi^{\mathbf{o}})^2 = 1 \ , \tag{7}$$

as $\mathcal{N} = \frac{\alpha}{2}(\mathbf{N}+1)$.

Searching the Dirac operator $\mathbf{D}$ by requiring the condition (b), i.e., $\{\gamma_\chi^{\mathbf{o}}, \mathbf{D}\} = 0$, we find

$$\mathbf{D} = \frac{i}{\ell\alpha}\gamma_\chi^{\mathbf{o}}\epsilon_{ijk}\sigma_i\mathbf{x}_j^{\mathbf{o}}\mathbf{x}_k \ . \tag{8}$$

Note that this Dirac operator is selfadjoint, $\mathbf{D}^\dagger = \mathbf{D}$.

Acting with this operator on the spinor $\Psi \in \Gamma\mathcal{A}_N$, we obtain

$$\mathbf{D}\Psi = \frac{i}{\ell}\gamma_\chi^{\mathbf{o}}\chi_i\mathbf{J}_i\Psi \ , \tag{9}$$

where

$$\mathbf{J}_i = \mathbf{L}_i + \frac{1}{2}\sigma_i \ , \tag{10}$$

and

$$\chi_i \equiv \epsilon_{ijk}\mathbf{x}_j\sigma_k \tag{11}$$

The angular momentum operator is defined by the adjoint action of $\mathbf{x}_i$:

$$\mathbf{L}_i\Psi \equiv \frac{1}{\alpha}[\mathbf{x}_i, \Psi] = \frac{1}{\alpha}(\mathbf{x}_i\Psi - \Psi\mathbf{x}_i) \tag{12}$$

If we replace the chirality operator $\gamma_\chi^{\mathbf{o}}$ by γ_χ in eq.(9), we obtain the Dirac operator given in ref.[16], since $\chi_i\mathbf{J}_i = \Sigma - \chi$, where $\Sigma = -i\epsilon_{ijk}\sigma_i\mathbf{x}_j\mathbf{L}_k$,

$\chi = \sigma_i \mathbf{x}_i$, and $\gamma_\chi(\Sigma - \chi)$ is the Dirac operator given in ref.[16] up to a normalization constant. The essential part of the Dirac operator in eq.(8) is the factor $\epsilon_{ijk}\sigma_i \mathbf{x}_j \mathbf{x}^{\mathbf{o}}_k$, which anticommutes with both $\gamma^{\mathbf{o}}_\chi$ and γ_χ.

The second condition of (b) concerning the commutative limit of the Dirac operator is also satisfied. If we take the commutative limit of each operator $\chi_i, \mathbf{J}_i, \gamma^{\mathbf{o}}_\chi$ in eq.(9), we obtain

$$\mathbf{D}_\infty = \frac{i}{\ell}\gamma_\chi \chi_i \mathbf{J}_i = \frac{i}{\ell^2}(\sigma_l x_l)\epsilon_{ijk} x_i \sigma_j (iK_k + \frac{1}{2}\sigma_k) = -\frac{1}{\ell}(i\sigma_i K_i + 1) \tag{13}$$

where x_i is the homogeneous coordinate of S^2 and K_i is the Killing vector. Therefore, in the commutative limit this Dirac operator is equivalent to the standard Dirac operator. For details see also ref.[18].

In order to establish Connes' triple we have to identify the Hilbert space.

The space of the fermions $\Psi \in \mathcal{A}_n \otimes M_2(\mathbb{C})$ defines the Hilbert space $\mathcal{H}_N$ with the norm

$$< \Psi|\Psi >= \mathrm{Tr}_{\mathcal{F}}(\Psi^\dagger \Psi) \tag{14}$$

where $\mathrm{Tr}_{\mathcal{F}}$ is the trace over the $(N+1)$ dimensional Hilbert space $\mathcal{F}_N$.

The dimension of the Hilbert space $\mathcal{H}_N$ is $2(N+1)^2$ and the trace over $\mathcal{H}_N$ is the trace over the spin suffices and over the $(N+1)^2$ dimensional space of the matrices. Since the Dirac operator is defined in the algebra $\mathcal{A}_N \otimes \mathcal{A}^{\mathbf{o}}_N$, in general the trace must be taken for operators of the form $\mathbf{ab^o}$, with $\mathbf{a}, \mathbf{b} \in \mathcal{A}_N$, and it is given by:

$$\mathrm{Tr}_{\mathcal{H}}\{\mathbf{ab^o}\} = \sum_{J=1}^{2(N+1)^2} < \Psi_J|\mathbf{ab^o}|\Psi_J >= 2\mathrm{Tr}_{\mathcal{F}}\{\mathbf{a}\}\mathrm{Tr}_{\mathcal{F}}\{\mathbf{b}\}\ , \tag{15}$$

where Ψ_J is an appropriate basis in $\mathcal{H}_N$. The factor 2 on the r.h.s. comes from the sum over spin suffices.

To examine the structure of the Hilbert space we compute the spectrum λ_j of the Dirac operator:

$$\mathbf{D}^2 \Psi_{jm} = \lambda_j^2 \Psi_{jm} \tag{16}$$

where Ψ_{jm} is a state with total angular momentum j, $\mathbf{J}^2\Psi_{jm} = j(j+1)\Psi_{jm}$, and $\mathbf{J}_3\Psi_{jm} = m\Psi_{jm}$. j and m are half integer and run $\frac{1}{2} \le j \le N + \frac{1}{2}$, $-j \le m \le j$. The spectrum is given by

$$\lambda_j^2 = (j + \frac{1}{2})^2[1 + \frac{1 - (j+\frac{1}{2})^2}{N(N+2)}]\ . \tag{17}$$

This spectrum is equivalent to the spectrum of the Dirac operator given in ref.[16]. It corresponds to the classical spectrum in the limit $N \to \infty$, except for the part with maximal angular momentum $j = N + \frac{1}{2}$. When the angular momentum takes its maximal value we see that $\lambda_{N+\frac{1}{2}} = 0$. This happens since there is no chiral pair for the spin $N + \frac{1}{2}$ state and therefore this part must be a zeromode for consistency. We can also confirm this property by computing the index, $\mathrm{Tr}_{\mathcal{H}}(\gamma_\chi)$. Since these zeromodes have no classical analogue, one way to treat this situation is to project them out from the Hilbert space. On the other hand, as we shall see, the contribution of the zeromodes in the integration is of order $\frac{1}{N}$ and thus their contribution vanishes in the commutative limit. Thus, for simplicity, we continue here working with the full Hilbert space $\mathcal{H}_N$, and we come back to this point in the discussion.

In this way, we obtain Connes' triple $(\mathcal{A}_N, \mathbf{D}, \mathcal{H}_N)$. We thus apply the construction of the differential calculus [3]. See also [19].

The exterior derivative can be defined for any element $\mathbf{a} \in \mathcal{A}_N$ as:

$$\pi(d\mathbf{a}) = [\mathbf{D}, \pi(\mathbf{a})] \tag{18}$$

where π is the representation of the algebra $\mathcal{A}_N$ in $\mathcal{H}_N$. In the following we do not distinguish between algebra element and its representation as far as it is obvious. Note that in our convention $(d\mathbf{a})^* = -d\mathbf{a}^*$. We define then the space of 1-forms Ω^1 by

$$\Omega^1 = \{\omega |\ \omega = \sum_i \mathbf{a}_i[\mathbf{D}, \mathbf{b}_i]\ ;\ \mathbf{a}_i, \mathbf{b}_i \in \mathcal{A}_N\}\ . \tag{19}$$

Thus, in the above construction the exterior derivative d is a map:

$$d \quad : \quad \mathcal{A}_N \ \to \ M_{\mathbb{C}}(2) \otimes (\mathcal{A}_N \otimes \mathcal{A}_N^{\mathbf{o}})\ . \tag{20}$$

To define the p-forms, one first introduces the universal differential algebra $\widetilde{\Omega}^* = \oplus \widetilde{\Omega}^p$, where a p-form is defined by the product

$$\widetilde{\omega_p} \equiv \mathbf{a}_0[\mathbf{D}, \mathbf{a}_1][\mathbf{D}, \mathbf{a}_2] \cdots [\mathbf{D}, \mathbf{a}_p]\ . \tag{21}$$

The exterior derivative is

$$d\widetilde{\omega_p} \equiv [\mathbf{D}, \mathbf{a}_0][\mathbf{D}, \mathbf{a}_1][\mathbf{D}, \mathbf{a}_2] \cdots [\mathbf{D}, \mathbf{a}_p]\ . \tag{22}$$

To obtain the graded differential algebra Ω^*_D, with Dirac operator $\mathbf{D}$ we have to divide out the differential ideal $\mathbf{J}+d\mathbf{J}$, $\mathbf{J}=ker(\pi)$, of the representation of the algebra [3] in the Hilbert space $\mathcal{H}_N$,

$$\Omega^*_D = \widetilde{\Omega}^*/(\mathbf{J}+d\mathbf{J}) \ . \tag{23}$$

The structure of this differential calculus with Dirac operator given in eq.(8) is discussed further in ref.[18].

4 Scalar Field

Let us apply this construction to define a scalar field on the fuzzy sphere. We denote the complex scalar field by $\Phi \in \mathcal{A}_N$, then its derivative is given by

$$d\Phi = [\mathbf{D},\Phi] = \frac{i}{\ell}\gamma^{\mathbf{o}}_{\chi}\epsilon_{ijk}\sigma_i\mathbf{x}^{\mathbf{o}}_j(\mathbf{L}_k\Phi) \ . \tag{24}$$

A natural choice for the action is

$$\mathbf{S} = \frac{1}{2(N+1)^2}\mathrm{Tr}_{\mathcal{H}}\{(d\Phi)^* d\Phi\} \ . \tag{25}$$

The above action can be evaluated by using the formula in eq.(15),

$$\mathbf{S} = \frac{-2}{3\alpha^2(N+1)}\mathrm{Tr}_{\mathcal{F}}\{[\mathbf{x}_i,\Phi^\dagger][\mathbf{x}_i,\Phi]\} \ , \tag{26}$$

Recalling that in the commutative limit, the trace $\frac{1}{N+1}\mathrm{Tr}_{\mathcal{F}}$ corresponds to the integration over the sphere:

$$\frac{1}{N+1}\mathrm{Tr}_{\mathcal{F}} \sim \int \frac{d\sigma^2}{4\pi\ell^2} \ , \tag{27}$$

the Langrangian is given by the 'integrand' of eq.(26). Using the angular momentum it can be written as

$$\mathcal{L} = \frac{2}{3}\sum_i |\mathbf{L}_i\Phi|^2 \ , \tag{28}$$

In the limit $N\to\infty$, the angular momentum is replaced by the Killing vector, and we obtain the standard Lagrangian of the scalar on the sphere.

From the equation of motion of the above scalar field, we obtain the Casimir operator as an Laplacian. Thus, the spectrum of the scalar boson on the noncommutative sphere coincides with the spectrum of the classical scalar boson, until the value where the spectrum of the scalar on the noncommutative sphere is truncated at its maximum angular momentum $l_{max} = N$, as expected [8].

5 Discussion

In this paper we have constructed the spectral triple using the new chirality operator $\gamma_\chi^{\mathbf{o}}$ on the fuzzy sphere. The differential calculus is formulated by applying Connes' construction. Since the algebra of the fuzzy sphere is represented by the operator in the Hilbert space $\mathcal{F}_N$, the simplest way to introduce the grading, for example, would have been to double the Hilbert space as $\mathcal{F}_N \oplus \mathcal{F}_N$. However, in the present approach, we first define the fermions as an $\mathcal{A}_N$-bimodule and then define the Hilbert space $\mathcal{H}_N$ of these fermions [3]. Then the algebra $\mathcal{A}_N$ is embedded into $\mathcal{A}_N \otimes \mathcal{A}_N^{\mathbf{o}}$ the elements of which are operators on $\mathcal{H}_N$. The Dirac operator as well as the chirality operator are thus constructed in terms of elements of $\mathcal{A}_N \otimes \mathcal{A}_N^{\mathbf{o}}$ and the exterior derivative is a map $d : \mathcal{A}_N \to M_{\mathbb{C}}(2) \otimes \mathcal{A}_N \otimes \mathcal{A}_N^{\mathbf{o}}$. The advantage of this approach is that we have a clear correspondence to the commutative case.

On the other hand, we have seen in section 3 that the Dirac operator possesses zeromodes in its spectrum which correspond to the state of maximal angular momentum and they have no classical analogue. In the context treated here, i.e. the scalar field on the sphere, the contribution of these zeromodes to the action is of order $1/N$ and therefore we obtain the standard classical theory in the commutative limit.

However, it is also possible to project out the zeromodes as follows: The spin $j = N + \frac{1}{2}$ state, $\Psi_{N+\frac{1}{2},m}$, can be constructed easily and the projection operator to the spin $N + \frac{1}{2}$ state is simply given by

$$\mathcal{P}_{N+\frac{1}{2}} = \sum_{k=0}^{2N+1} |\Psi_{N+\frac{1}{2},N+\frac{1}{2}-k} >< \Psi_{N+\frac{1}{2},N+\frac{1}{2}-k}| \ . \tag{29}$$

[3]Such a strategy is also taken in refs.[13, 15].

By using this projection operator we may obtain the Hilbert space $\mathcal{H}'_N$ which has no zeromodes. However, in such a case we must reconsider the whole operator algebra under this projection, since the algebra $\mathcal{A}_N$ is not represented in the Hilbert space $\mathcal{H}'_N$. Such a projection may also be avoidable by considering the supersymmetry [13].

The algebra treated here is simply a matrix algebra and by construction, we obtain the algebra of the spherical harmonics in the limit $N \to \infty$, which is the reason for calling it the algebra of the fuzzy sphere. As an algebra, $\mathcal{A}_N$ is equivalent to the algebra $M_{\mathbb{C}}(N+1)$ of the nonsingular $(N+1) \times (N+1)$ matrices for given finite N.

The same algebra can be obtained by applying Berezin's quantization for the Poisson algebra on the sphere. By the same procedure, we can also define the fuzzy torus and fuzzy Riemann surfaces in general, since Berezin's quantization procedure can be applied to any Kähler manifold [20, 21, 22, 23, 24]. As a result we obtain a noncommutative algebra which is the quantization of the function algebra over the corresponding manifold (considering the complex manifold as a Poisson manifold, i.e. as a phase space). For example the algebra of the fuzzy torus (or the noncommutative torus) is generated by two element $\mathbf{S}, \mathbf{T}$ with the commutation relation

$$\mathbf{S}\mathbf{T} = q\mathbf{T}\mathbf{S} \tag{30}$$

For each case, we obtain a matrix algebra.

Therefore, we encounter the question: if we are given a matrix algebra, how do we find the corresponding classical geometry. To our knowledge, there are no general criteria to answer this question. It depends on the way how the limit is taken.

When we define the Dirac operator in this paper and also in ref.[16], we have required that the commutative limit of this operator coincides with the standard Dirac operator on the sphere. However, as we disscussed above, we also have to specify the condition of how the limit has to be taken. One of the criteria to specify the commutative limit, including the differential operator, is the boundedness of the operator $[\mathbf{D}, \mathbf{a}]$ for $\mathbf{a} \in \mathcal{A}_N$. In Connes' construction, the boundedness of the commutator $[\mathbf{D}, \mathbf{a}]$ is required. It is also clear that if this commutator is not bounded, then the limit operator is not an operator in the algebra of the spherical harmonics. So this requirement is necessary to specify the limit as a sphere. Furthermore, the condition of

boundedness should be also sufficient to specify the topology of the Riemann surface obtained in the limit $N \to \infty$ of the matrix algebra $M_{\mathbb{C}}(N)$, since in the present case the boundedness of the commutator $[\mathbf{D}, \mathbf{a}]$ defines a finite Poisson bracket of the algebra elements with the generators of the algebra $\mathbf{x}_i$.

As we have seen, the definition of the Dirac operator has an ambiguity due to the operator ordering. In this paper we discussed only the fuzzy sphere. However, since it is possible to discuss any fuzzy Riemann surface with the Berezin-Toepliz quantization of the corresponding Riemann surface[20, 21, 22, 23], we also want to have a common description of the differential calculus on fuzzy Riemann surfaces which include the case of higher genus. Such a requirement may restrict the Dirac operator. Another interesting aspect to consider all Riemann surfaces within a common framework is that a field theory on the fuzzy Riemann surface may be considered as a fuzzy string, the world sheet of which is discribed by a fuzzy Riemann surface.

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